

Exercise sheet #5

Problem 1. Imagine a world where the force of interaction between two charges is:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \left(1 + \frac{r}{\lambda}\right) e^{(-r/\lambda)} \hat{r}$$

where λ is a constant with dimensions of length and $\lambda \gg 0$ (i.e. it's really large so the correction to Coulomb's law given by the equation above is really small). Assuming that the principle of superposition still holds:

- (a) What is the electric field of a charge distribution ρ ? i.e. what is the equation that will replace $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{r^2} \hat{r} dV$
- (b) Does the electric field admit a scalar potential? Explain briefly your conclusion. Don't give a formal proof, rather a convincing argument.
- (c) Find the potential ϕ for a point charge q (If your answer to (b) was no better go back and check it. Use ∞ as your reference point.
- (d) Using the expression for $\vec{E}$ and ϕ you found. Show that for a point charge q at the origin:

$$\oint_S \vec{E} \cdot d\vec{a} + \frac{1}{\lambda^2} \int_V \phi dV = \frac{1}{\epsilon_0} q$$

where S is the surface and V the volume of any sphere centered at q .

- (e) Show that this result generalizes to

$$\oint_S \vec{E} \cdot d\vec{a} + \frac{1}{\lambda^2} \int_V \phi dV = \frac{1}{\epsilon_0} Q_{enc}$$

for any charge distribution.

- (f) Show that, in contrast to our world, some of the charge on a conductor distributes itself **uniformly** over the volume, with the remainder on the surface. [Hint: First make an argument to show $\vec{E}$ is still zero, inside a conductor.]

Solution: (a) $\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{r^2} \left(1 + \frac{r}{\lambda}\right) e^{-r/\lambda} dV$.

- (b) Yes. The field of a point charge at the origin is radial and symmetric, so $\nabla \times \vec{E} = 0$, and hence this is also true (by superposition) for any collection of charges.

- (c)

$$\begin{aligned} \phi &= - \int_{\infty}^r \vec{E} \cdot d\vec{s} = - \frac{1}{4\pi\epsilon_0} q \int_{\infty}^r \frac{1}{r'^2} \left(1 + \frac{r'}{\lambda}\right) e^{-r'/\lambda} dr' \\ &= \frac{1}{4\pi\epsilon_0} q \int_r^{\infty} \frac{1}{r'^2} \left(1 + \frac{r'}{\lambda}\right) e^{-r'/\lambda} dr' = \frac{q}{4\pi\epsilon_0} \left\{ \int_r^{\infty} \frac{1}{r'^2} e^{-r'/\lambda} dr' + \frac{1}{\lambda} \int_r^{\infty} \frac{1}{r'} e^{-r'/\lambda} dr' \right\} \end{aligned}$$

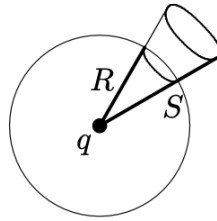
Now $\int \frac{1}{r^2} e^{-r/\lambda} dr = -\frac{e^{-r/\lambda}}{r} - \frac{1}{\lambda} \int \frac{e^{-r/\lambda}}{r} dr \leftarrow$ exactly right to kill the last term. Therefore

$$\phi(r) = \frac{q}{4\pi\epsilon_0} \left\{ -\frac{e^{-r/\lambda}}{r} \Big|_r^{\infty} \right\} = \frac{q}{4\pi\epsilon_0} \frac{e^{-r/\lambda}}{r}$$

(d)

$$\begin{aligned}
\oint_S \vec{E} \cdot d\vec{a} &= \frac{1}{4\pi\epsilon_0} q \frac{1}{R^2} \left(1 + \frac{R}{\lambda}\right) e^{-R/\lambda} 4\pi R^2 = \frac{q}{\epsilon_0} \left(1 + \frac{R}{\lambda}\right) e^{-R/\lambda}. \\
\int_V \phi dV &= \frac{q}{4\pi\epsilon_0} \int_0^R \frac{e^{-r/\lambda}}{r} r^2 4\pi dr = \frac{q}{\epsilon_0} \int_0^R r e^{-r/\lambda} dr = \frac{q}{\epsilon_0} \left[\frac{e^{-r/\lambda}}{(1/\lambda)^2} \left(-\frac{r}{\lambda} - 1\right) \right]_0^R \\
&= \lambda^2 \frac{q}{\epsilon_0} \left\{ -e^{-R/\lambda} \left(1 + \frac{R}{\lambda}\right) + 1 \right\}. \\
\therefore \oint_S \vec{E} \cdot d\vec{a} + \frac{1}{\lambda^2} \int_V \phi dV &= \frac{q}{\epsilon_0} \left\{ \left(1 + \frac{R}{\lambda}\right) e^{-R/\lambda} - \left(1 + \frac{R}{\lambda}\right) e^{-R/\lambda} + 1 \right\} = \frac{q}{\epsilon_0}.
\end{aligned}$$

(e) Does the result hold for a nonspherical surface? Suppose we make a “dent” in the sphere—pushing a patch (area $R^2 \sin \theta d\theta d\phi$) from radius R out to radius S (area $S^2 \sin \theta d\theta d\phi$).



$$\begin{aligned}
\Delta \oint \vec{E} \cdot d\vec{a} &= \frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{S^2} \left(1 + \frac{S}{\lambda}\right) e^{-S/\lambda} (S^2 \sin \theta d\theta d\phi) - \frac{1}{R^2} \left(1 + \frac{R}{\lambda}\right) e^{-R/\lambda} (R^2 \sin \theta d\theta d\phi) \right\} \\
&= \frac{q}{4\pi\epsilon_0} \left[\left(1 + \frac{S}{\lambda}\right) e^{-S/\lambda} - \left(1 + \frac{R}{\lambda}\right) e^{-R/\lambda} \right] \sin \theta d\theta d\phi \\
\Delta \frac{1}{\lambda^2} \int \phi dV &= \frac{1}{\lambda^2} \frac{q}{4\pi\epsilon_0} \int \frac{e^{-r/\lambda}}{r} r^2 \sin \theta dr d\theta d\phi = \frac{1}{\lambda^2} \frac{q}{4\pi\epsilon_0} \sin \theta d\theta d\phi \int_R^S r e^{-r/\lambda} dr \\
&= -\frac{q}{4\pi\epsilon_0} \sin \theta d\theta d\phi \left(e^{-r/\lambda} \left(1 + \frac{r}{\lambda}\right) \right) \Big|_R^S \\
&= -\frac{q}{4\pi\epsilon_0} \left[\left(1 + \frac{S}{\lambda}\right) e^{-S/\lambda} - \left(1 + \frac{R}{\lambda}\right) e^{-R/\lambda} \right] \sin \theta d\theta d\phi
\end{aligned}$$

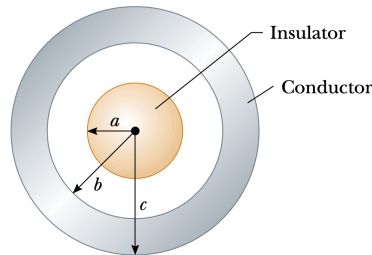
So the change in $\frac{1}{\lambda^2} \int \phi dV$ exactly compensates for the change in $\oint \vec{E} \cdot d\vec{a}$, and we get $\frac{1}{\epsilon_0} q$ for the total using the dented sphere, just as we did with the perfect sphere. Any closed surface can be built up by successive distortions of the sphere, so the result holds for all shapes. By superposition, if there are many charges inside, the total is $\frac{1}{\epsilon_0} Q_{\text{enc}}$. Charges outside do not contribute (in the argument above we found that for this volume $\oint \vec{E} \cdot d\vec{a} + \frac{1}{\lambda^2} \int \phi dV = 0$ and, again, the sum is not changed by distortions of the surface, as long as q remains outside. So this new “Gauss’s Law” holds for any charge configuration.

(f) If you first express this new “Gauss’s law” in differential form, it reads: $\nabla \cdot \vec{E} + \frac{1}{\lambda^2} \phi = \frac{1}{\epsilon_0} \rho$. $\vec{E}$ is zero, inside a conductor (otherwise charge would move, and in such a direction as to cancel the field), thus ϕ is constant (inside), and hence ρ is **uniform**, throughout the volume. Any “extra” charge must reside on the surface. (The fraction at the surface depends on λ , and on the shape of the conductor.)

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Problem 2. A solid, insulating sphere of radius a has a uniform charge density ρ and a total charge Q . Concentric with this sphere is an uncharged, conducting hollow sphere whose inner and outer radii are b and c , as shown in the figure below.

- (a) Find the magnitude of the electric field in the regions $r < a$, $a < r < b$, $b < r < c$, and $r > c$.
- (b) Determine the induced charge per unit area on the inner and outer surfaces of the hollow sphere.



Solution: (a) From Gauss' law $\oint \vec{E} \cdot d\vec{A} = E(4\pi r^2) = \frac{q_{\text{in}}}{\epsilon_0}$. For $r < a$, $q_{\text{in}} = \rho \frac{4}{3}\pi r^3$, so $E = \frac{\rho r}{3\epsilon_0}$. For $a < r < b$ and $c < r$, $q_{\text{in}} = Q$, so $E = \frac{Q}{4\pi\epsilon_0 r^2}$. For $b \leq r \leq c$, $E = 0$, since $E = 0$ inside a conductor.

- (b) Let q_1 be the induced charge on the inner surface of the hollow sphere. Since $E = 0$ inside the conductor, the total charge enclosed by a spherical surface of radius $b \leq r \leq c$ must be zero. Therefore, $q_1 + Q = 0$ and $\sigma_1 = \frac{q_1}{4\pi b^2} = -\frac{Q}{4\pi b^2}$. Let q_2 be the induced charge on the outside surface of the hollow sphere. Since the hollow sphere is uncharged, we require $q_1 + q_2 = 0$ and $\sigma_2 = \frac{q_2}{4\pi c^2} = \frac{Q}{4\pi c^2}$.

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Problem 3. If a point charge is located outside a hollow conducting shell, there is an electric field outside, but no electric field inside. On the other hand, if a point charge is located inside a hollow conducting shell, there is an electric field both inside and outside (although the external field would be zero in the special case where the shell happened to have charge exactly equal and opposite to the point charge). The situation is therefore not symmetric with respect to inside and outside. Explain why this is the case, by considering where electric field lines can begin and end.

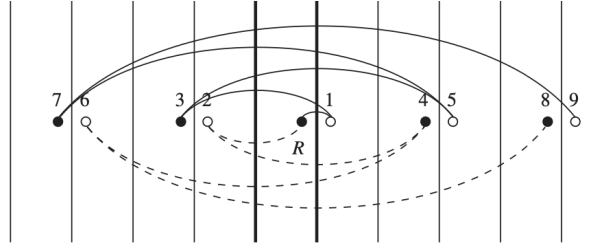
Solution: Electrostatic field lines can begin and end only at charges or at infinity. Also, there can be no closed loops since $\nabla \times \vec{E} = 0$. If the point charge is located outside the shell, the field lines can have their ends at the point charge, the shell, or infinity. There can't be any field lines inside the shell because they would have to start at one point on the shell and end at another. This would imply a nonzero potential difference between these two points, contradicting the fact that all points on the shell have the same potential.

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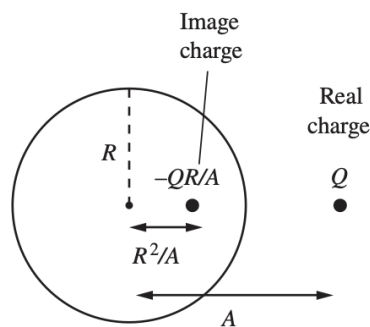
Problem 4. A point charge q is located between two parallel infinite conducting planes, a distance d from one and $l - d$ from the other. Where should image charges be located so that the electric field is everywhere perpendicular to the planes?

Solution: In the figure below the two given planes are indicated by the bold lines, and the given real charge is labeled R. It turns out that we will need an infinite number of image charges, as shown. Solid dots are positive, hollow dots are negative (assuming the given real charge is positive). The reason for all these image charges is the following. In order to have the E field be perpendicular to the right plane, we need the image charge labeled 1. And likewise, in order to have E be perpendicular to the left plane, we need image charge 2. So far, we just have two copies of the one-plane setup. However, image charge 1 ruins the orthogonality of the field with the left plane, so we need image charge 3 to remedy this. Likewise, image charge 2 ruins the orthogonality of the field with the right plane, so we need image charge 4 to remedy this. But then we need image charges 5 and 6 to remedy the effects of 3 and 4, respectively. And so on. The effects of the charges far away are small, so the process converges. That is, if we have 1000 such charges, the field will be very nearly perpendicular everywhere to the two given

planes. If you want, you can group the charges into two sets – the odds and evens, as indicated by the connecting lines in the figure. Each odd charge corrects the effect of the previous odd charge, with respect to alternating planes. Likewise for the evens. In the special case where the given real charge is located midway between the two planes, all the image charges are similarly located midway between the (imaginary) planes in the figure. So the net force on the given charge is zero, as it should be. $\square$



- Problem 5.** (a) A point charge $-q$ is located at $x = a$, and a point charge Q is located at $x = A$. Show that the locus of points with $\phi = 0$ is a circle in the xy plane (and hence a spherical shell in space).
- (b) What is the relation among q, Q, a , and A so that the center of the circle is located at $x = 0$?
- (c) Assuming that the relation you found in part (b) holds, what is the radius of the circle in terms of a and A ?
- (d) Explain why the previous results imply the following statement: if a charge Q is externally located a distance $A > R$ from the center of a grounded conducting spherical shell with radius R , then the external field due to the shell is the same as the field of an image point charge $-q = -QR/A$ located a distance $a = R^2/A$ from the center of the shell (See Figure below). The total external field is the sum of this field plus the field from Q . (The internal field is zero, by the uniqueness theorem.)



Solution: (a) The potential at an arbitrary point in the xy plane is (ignoring the $1/4\pi\epsilon_0$)

$$\phi = \frac{Q}{\sqrt{(x-A)^2 + y^2}} - \frac{q}{\sqrt{(x-a)^2 + y^2}}$$

Setting this equal to zero, putting one term on either side of the equation, and squaring gives:

$$Q^2 (x^2 - 2ax + a^2 + y^2) = q^2 (x^2 - 2Ax + A^2 + y^2) \quad (1)$$

Since the coefficients of x^2 and y^2 are equal, this equation describes a circle. More precisely, the equation can be written in the form of $x^2 + y^2 - 2Bx = C$, which in turn can be written as $(x-B)^2 + y^2 = C + B^2$, by completing the square. This equation describes a circle with its center at $(B, 0)$ and with radius $\sqrt{C + B^2}$.

(b) Expanding Eq. 1 gives:

$$(Q^2 - q^2)x^2 + (Q^2 - q^2)y^2 - 2(Q^2a - q^2A)x = q^2A^2 - Q^2a^2 \quad (2)$$

The center of the circle is located at $x = 0$ if the coefficient of x is zero, that is, if $Q^2a = q^2A$.

Alternatively, we can work in terms of the angle θ shown in the figure below. Using the law of cosines to determine the distances from a point P on the circle to the two charges (assuming the center is located at $x = 0$), we see that the potential at P is zero if:

$$\begin{aligned} \frac{Q}{\sqrt{R^2 + A^2 - 2RA \cos \theta}} &= \frac{q}{\sqrt{R^2 + a^2 - 2Ra \cos \theta}} \\ \implies Q^2(R^2 + a^2 - 2Ra \cos \theta) &= q^2(R^2 + A^2 - 2RA \cos \theta) \end{aligned} \quad (3)$$

If this equation is to be true for all values of θ , then the coefficient of $\cos \theta$ must be the same on both sides. This yields $Q^2a = q^2A$, as above.

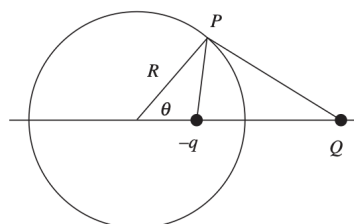
(c) If $Q^2a = q^2A$, then dividing Eq. 2 by $Q^2 - q^2$ tells us that the radius of the circle is given by

$$R^2 = \frac{q^2A^2 - Q^2a^2}{Q^2 - q^2} = \frac{(Q^2a/A)A^2 - Q^2a^2}{Q^2 - (Q^2a/A)} = aA$$

The radius is therefore the geometric mean of the distances from the two charges to the center of the circle. Alternatively, we can work with the angle θ shown in the figure below. If $Q^2a = q^2A$, then Eq.3 gives:

$$\begin{aligned} Q^2(R^2 + a^2) &= q^2(R^2 + A^2) \\ \implies Q^2(R^2 + a^2) &= (Q^2a/A)(R^2 + A^2) \\ \implies A(R^2 + a^2) &= a(R^2 + A^2) \\ \implies R^2(A - a) &= aA(A - a) \\ \implies R^2 &= aA \end{aligned}$$

(d) Having derived $R^2 = aA$, we can eliminate a from the relation $Q^2a = q^2A$ to obtain $Q^2(R^2/A) = q^2A \implies q = QR/A$. Putting all the results together, we see that if we have a charge Q at $x = A$ and a charge $-q = -QR/A$ at $x = a = R^2/A$, then the entire surface of the sphere of radius R centered at the origin will be at zero potential. But this is exactly the boundary condition for a grounded conducting sphere. The uniqueness theorem therefore tells us that the two setups (point charge outside grounded conducting sphere, and point charge near image charge) have exactly the same field in the exterior of the sphere. (This reasoning doesn't apply to the interior, because the setups are different there; one contains an image charge, the other doesn't. The uniqueness theorem requires the same charge distribution in the relevant region in both setups.) The results for this problem look a little cleaner if we let $A = nR$, where n is a numerical factor. The image charge then has the value $-q = -Q/n$ and is located at radius R/n .



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Problem 6. Find the capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b .



Solution: Say the charge on the inner cylinder is Q , for a length L . The field is given by Gauss's law: $\int \vec{E} \cdot d\vec{a} = E \cdot 2\pi s \cdot L = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0} Q \Rightarrow \vec{E} = \frac{Q}{2\pi\epsilon_0 L} \frac{1}{s} \hat{s}$. Potential difference between the cylinders is

$$\phi(b) - \phi(a) = - \int_a^b \vec{E} \cdot d\vec{l} = - \frac{Q}{2\pi\epsilon_0 L} \int_a^b \frac{1}{s} ds = - \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$$

As set up here, a is at the higher potential, so $V = \phi(a) - \phi(b) = \frac{Q}{2\pi\epsilon_0 L} \ln\left(\frac{b}{a}\right)$. $C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$, so capacitance per unit length is $\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$. $\square$